DEFINITE AND SEMIDEFINITE QUADRATIC FORMS[1]

By Gerard Debreu

The problem of maximization (or minimization) of a function $f(\xi)$ of a vector ξ, possibly constrained by m relations $g_j(\xi) = 0$ $(j = 1, \cdots, m)$, is outstanding in the classical economic theories of the consumer (who maximizes utility subject to a budgetary constraint) and of the firm (which maximizes profit subject to technological and other constraints). The calculus treatment of this question (see for example [**5**]) leads to a scrutiny of the conditions that a quadratic form be definite or semidefinite, with or without linear constraints. These conditions have also found an important application in studies of the equilibrium stability of a multiple-market economy. However, although they were frequently used, it is rather difficult to find short and complete proofs in the literature. Original proofs are given here in a unified treatment of the subject.

A, B, x are matrices of orders[2] $n \cdot n$, $n \cdot m$, $n \cdot 1$. M being a matrix, M_{pq} is obtained from M by keeping only the elements in the first p rows and the first q columns; M_p stands for M_{pp}; when M is square, $|M|$ is its determinant. Primed letters denote transposes.

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m_{ij} is the ith row, jth column element of M; x_i stands for x_{i1}; L_r is a linear form in the variables x_i, $r \leqslant i \leqslant n$, whose coefficient of x_r is unity. As a matter of convention, $|A_0| = 1$.

Theorem 1: *Let A be symmetric. $x'Ax = \sum_{r=1}^{n} (|A_r| / |A_{r-1}|)(L_r)^2$ if and only if $|A_r| \neq 0$ for $r = 1, \cdots, n-1$.*

Necessity:[3] Obvious. *Sufficiency:* Perform on the quadratic form $Q_1(x_1, \cdots, x_n) = x'Ax$ a standard decomposition into squares and assume that after the first $r - 1$ steps Q_1 has been written in the form

$$(1) \qquad Q_1(x_1, \cdots, x_n) = \sum_{i,j} a_{ij}x_i x_j = c_1(L_1)^2 + \cdots + c_{r-1}(L_{r-1})^2 + Q_r(x_r, \cdots, x_n),$$

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[2] A matrix of order $n \cdot m$ is one having n rows and m columns.

[3] In all the equivalence statements of the form "U if and only if V," the *necessity* part is understood to be "U implies V;" the *sufficiency* part, "V implies U."

where $c_i \neq 0$ for $i = 1, \cdots, r-1$, and Q_r is a quadratic form in the variables x_i, $r \leqslant i \leqslant n$. We wish to find the coefficient c_r of x_r^2 in Q_r.

Set $x_i = 0$ for $i = r+1, \cdots, n$, and then derive from (1) the r identities

$$\frac{1}{2}\frac{\partial Q_1}{\partial x_i} = \sum_{j=1}^{r} a_{ij}x_j = \sum_{j=1}^{i} c_j L_j \frac{\partial L_j}{\partial x_i} \qquad (i = 1, \cdots, r-1),$$

$$\frac{1}{2}\frac{\partial Q_1}{\partial x_r} = \sum_{j=1}^{r} a_{rj}x_j = \sum_{j=1}^{r-1} c_j L_j \frac{\partial L_j}{\partial x_r} + c_r x_r.$$

Make $L_j = 0$ for $j = 1, \cdots, r-1$. This system has a nonzero solution and therefore also

$$\sum_{j=1}^{r} a_{ij}x_j = 0 \qquad (i = 1, \cdots, r-1),$$

$$\sum_{j=1}^{r} a_{rj}x_j - c_r x_r = 0,$$

i.e.,

$$\begin{vmatrix} A_{r-1} & a_{ir} \\ a_{rj} & a_{rr} - c_r \end{vmatrix} = 0 \quad \text{or} \quad c_r = \frac{|A_r|}{|A_{r-1}|}.$$

Since $|A_r| \neq 0$, it is possible to perform the rth step.

THEOREM 2: *Let A be symmetric. $x'Ax > 0$ (resp. < 0) for every $x \neq 0$ if and only if $|A_r| > 0$ [resp. $(-1)^r |A_r| > 0$] for $r = 1, \cdots, n$.*

If $Ax = 0$ had a nonzero solution x_0, one would have $x_0' Ax_0 = 0$, and the quadratic form would not be definite. It is therefore necessary that $|A| \neq 0$ and more generally $|A_r| \neq 0$ for $r = 1, \cdots, n$ [set $x_{r+1}, \cdots, x_n$ all equal to zero; the quadratic form $Q_1(x_1, \cdots, x_r, 0, \cdots, 0)$ must also be definite].

A straightforward application of Theorem 1 then proves the statement.

2. QUADRATIC FORMS DEFINITE UNDER LINEAR CONSTRAINTS

THEOREM 3: *$x'Ax > 0$ (resp. < 0) for every $x \neq 0$ such that $B'x = 0$ if and only if there exists a number λ such that $x'Ax + \lambda x'BB'x$ is a positive (resp. negative) definite quadratic form.*

Sufficiency: Obvious. *Necessity:* The function $\theta(x) = -(x'Ax/x'BB'x)$ is continuous on the set $\{x \mid x'x = 1 \text{ and } B'x \neq 0\}$, and tends to $-\infty$ (resp. $+\infty$) whenever x tends to a boundary point; it has therefore a finite maximum (resp. minimum) λ^*. Any $\lambda > \lambda^*$ (resp. $< \lambda^*$) has the desired property.[4]

[4] As $B'x = 0$ is *equivalent* to $x'BB'x = 0$, "under the constraint $B'x = 0$, $x'Ax$ is positive definite" is equivalent to "under the constraint $x'BB'x = 0$, $x'Ax$ has a minimum reached at, and only at, $x = 0$." The Lagrange multiplier technique could thus hint at the existence of a number λ such that $x'Ax + \lambda x'BB'x$ has a minimum reached at, and only at, $x = 0$.

LEMMA: $|A + \lambda BB'|$ *is a polynomial in* λ *whose term of highest order (possibly null) is* $(-1)^m \begin{vmatrix} A & B \\ B' & 0_m \end{vmatrix} \lambda^m$.

From

$$\begin{bmatrix} A & \lambda B \\ B' & -I_m \end{bmatrix} \begin{bmatrix} I_n & 0_{nm} \\ B' & I_m \end{bmatrix} = \begin{bmatrix} A + \lambda BB' & \lambda B \\ 0_{mn} & -I_m \end{bmatrix}$$

follows $\begin{vmatrix} A & \lambda B \\ B' & -I_m \end{vmatrix} = (-1)^m |A + \lambda BB'|$. In the development of the left-hand determinant a term contains the highest possible power of λ if in every one of the last m columns one takes an element of λB. Such terms are unaffected if $-I_m$ is replaced by any other $m \cdot m$ matrix: take 0_m.

THEOREM 4: *Let* A *be symmetric and* $|B_m|$ *be different from zero.* $x'Ax > 0$ *for every* $x \neq 0$ *such that* $B'x = 0$ *if and only if* $(-1)^m \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0$ *for* $r = m + 1, \cdots, n$.[5]

Necessity: Consider the equations $\begin{cases} Ax + By = 0 \\ B'x \qquad = 0 \end{cases}$ where y is a $m \cdot 1$ matrix. A solution $\begin{bmatrix} x \\ y \end{bmatrix}$ is such that $x'Ax + x'By = 0$, i.e., $x'Ax = 0$. This must imply $x = 0$, therefore $By = 0$, and, since $|B_m| \neq 0$, $y = 0$. The system must have no other solution than 0, i.e., $\begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} \neq 0$.

From Theorem 3 and Theorem 2, for every $\lambda > \lambda^*$ one must have $|A + \lambda BB'| > 0$.

From the lemma one must have $(-1)^m \begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} > 0$.

This argument can be made for any r, $m \leqslant r \leqslant n$.

Sufficiency: I shall prove that the coefficient of the term of highest order in λ of $|A_r + \lambda B_{rm} B'_{rm}|$ is positive for all $r = 1, \cdots, n$. It will therefore be possible to choose λ large enough to make these n leading minors positive and consequently $[A + \lambda BB']$ positive definite (Theorem 2).

[5] The symbol B_m implies by itself that $m \leqslant n$. The second half of this statement may be expressed in another and perhaps more convenient form: "... *if and only if every northwest principal minor of* $\begin{bmatrix} 0 & B' \\ B & A \end{bmatrix}$ *of order larger than* $2m$ *has the sign of* $(-1)^m$." A similar remark can be made for the statement of Theorem 5.

(a) If $r > m$, it is true by assumption.
(b) If $r \leqslant m$, write (cf. proof of the lemma)

$$(-1)^m \,|\, A_r + \lambda B_{rm} B'_{rm} \,| = \begin{vmatrix} A_r & \lambda B_{rm} \\ B'_{rm} & -I_m \end{vmatrix} = \lambda^r \begin{vmatrix} (1/\lambda) A_r & B_{rm} \\ B'_{rm} & -I_m \end{vmatrix}.$$

When λ tends to $\pm\infty$, $(1/\lambda)A_r$ tends to 0_r and therefore the (possibly null) term of highest order of the left-hand polynominal in λ is $\lambda^r \begin{vmatrix} 0_r & B_{rm} \\ B'_{rm} & -I_m \end{vmatrix}$. The quadratic form $-y'y$ is negative definite; it is *a fortiori* negative definite under the constraint $B_{rm}y = 0$. Moreover, not all $r \cdot r$ minors of B_{rm} can vanish since $|\, B_m \,| \neq 0$. Therefore, according to the proof of necessity in Theorem 5 (which is identical to that in Theorem 4), $(-1)^m \begin{vmatrix} 0_r & B_{rm} \\ B'_{rm} & -I_m \end{vmatrix} > 0$.

A similar argument proves

THEOREM 5: *Let A be symmetric and $|\, B_m \,|$ be different from zero. $x'Ax < 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if* $(-1)^r \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0$ *for $r = m + 1, \cdots, n$.*

3. SEMIDEFINITE QUADRATIC FORMS

Let π denote a permutation of the first n integers; A^π, the matrix obtained from A by performing the permutation π on its rows and on its columns; B^π, the matrix obtained from B by performing the permutation π on its rows.

THEOREM 6: *$x'Ax \geqslant 0$ (resp. $\leqslant 0$) for every x if and only if $x'Ax + \alpha x'x > 0$ (resp. < 0) for every $x \neq 0$ and every $\alpha > 0$ (resp. < 0).*

Necessity: Obvious. *Sufficiency:* Obvious by a continuity argument.

THEOREM 7: *Let A be symmetric. $x'Ax \geqslant 0$ (resp. $\leqslant 0$) for every x if and only if $|\, A_r^\pi \,| \geqslant 0$ [resp. $(-1)^r \,|\, A_r^\pi \,| \geqslant 0$] for all $r = 1, \cdots, n$ and all π.*

Necessity: From Theorem 6 and Theorem 2, $|\, A_r^\pi + \alpha I_r \,| > 0$, for $r = 1, \cdots, n$, $\alpha > 0$, and any π. This implies $|\, A_r^\pi \,| \geqslant 0$ for $r = 1, \cdots, n$ and any π.

Sufficiency: $|\, A_r + \alpha I_r \,| = \alpha^r + \sum_{i=0}^{r-1} \alpha^i S_{r-i}^r$ where S_i^r is the sum of all the principal minors of A_r of order i. From the assumption, every S_i^r is nonnegative and, if α is positive, the right-hand member is positive. This implies (Theorem 2) that the quadratic form $x'Ax + \alpha x'x$ is positive definite for any $\alpha > 0$. An application of Theorem 6 yields the result.

The result for negative forms is proved by a transposed argument.

4. QUADRATIC FORMS SEMIDEFINITE UNDER LINEAR CONSTRAINTS

The same techniques yield

THEOREM 8: *$x'Ax \geqslant 0$ (resp. $\leqslant 0$) for every x such that $B'x = 0$ if and only if $x'Ax + \alpha x'x > 0$ (resp. < 0) for every $\alpha > 0$* (resp. < 0) *and every $x \neq 0$ such that $B'x = 0$.*

THEOREM 9: *Let A be symmetric and $|B_m|$ be different from zero. $x'Ax \geqslant 0$ for every x such that $B'x = 0$ if and only if $(-1)^m \begin{vmatrix} A_r^\pi & B_{rm}^\pi \\ B_{rm}^{\pi\prime} & 0 \end{vmatrix} \geqslant 0$, for $r = m + 1, \cdots, n$ and any π.*

[Note that now the term of highest order in α in the development of $(-1)^m \begin{vmatrix} A_r + \alpha I_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix}$ is $\alpha^{r-m} \sum |\tilde{B}_m|^2$ where $\tilde{B}_m$ is any $m \cdot m$ submatrix of B_{rm} whose rows are in the natural order.]

THEOREM 10: *Let A be symmetric and $|B_m|$ be different from zero. $x'Ax \leqslant 0$ for every x such that $B'x = 0$ if and only if $(-1)^r \begin{vmatrix} A_r^\pi & B_{rm}^\pi \\ B_{rm}^{\pi\prime} & 0 \end{vmatrix} \geqslant 0$, for $r = m + 1, \cdots,$ and any π.*

5. HISTORICAL NOTE

The idea of the decomposition into squares of Theorem 1 goes back to Lagrange [**4**] who performed it, however, only in the cases $n = 2, 3$ and did not conjecture the general form of the coefficient $|A_r| \, |A_{r-1}|$. This decomposition was given for the first time in all its generality by F. Brioschi [**1**].

The form of the conditions used in the statements of Theorems 4 and 5 appeared for the first time in a paper by H. Hotelling [**3**] (in the case $m = 1$). The first complete statement and proof of Theorems 4 and 5 are due to H. B. Mann [**6**].[6]

The method used by Hotelling [**3**], and Samuelson [**8**, pp. 376–378], which goes back to Richelot [**7**], consists in studying the roots of the equation in λ, $\begin{vmatrix} A - \lambda I_n & B \\ B' & 0 \end{vmatrix} = 0$. It easily proves Theorems 9 and

[6] Other proofs, rather intricate, were given for Theorem 4 by S. N. Afriat ("The Quadratic Form Positive Definite on a Linear Manifold," *Proceedings of the Cambridge Philosophical Society*, Vol. 47, January, 1951, Part I, pp. 1–6), for Theorems 4 and 5 by J. Seitz ("Note sur un problème fondamental de la théorie de l'équilibre économique," *Aktuárské Vědy*, Vol. 8, 1949, pp. 137–144), and for a particular case of Theorem 5 ($m = 1$, all elements of B equal to unity) by M. Allais (*A la recherche d'une discipline économique*, Vol. I, Paris: Ateliers Industria, 1943, Annexes, pp. 25–28).

10 and the *necessity* of the conditions given in Theorems 4 and 5, but it does not prove that it is sufficient that the *leading* minors of $\begin{vmatrix} A & B \\ B' & 0 \end{vmatrix}$ be of the proper signs.

Similarly, the study [8, pp. 370–375] of the roots of $|A - \lambda I| = 0$ easily proves Theorem 7 and the necessity of the conditions of Theorem 2; it cannot prove their sufficiency without additional steps (see [2]).

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